

WOLFF'S PROBLEM OF IDEALS IN THE MULTIPLIER ALGEBRA ON DIRICHLET SPACE

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ABSTRACT. We establish an analogue of Wolff's theorem on ideals in $H^\infty(\mathbb{D})$ for the multiplier algebra of Dirichlet space.

In 1962 Carleson [C] proved his famous ‘‘Corona theorem’’ characterizing when a finitely generated ideal in $H^\infty(\mathbb{D})$ is actually all of $H^\infty(\mathbb{D})$. Independently, Rosenblum [R], Tolokonnikov [To], and Uchiyama gave an infinite version of Carleson's work on $H^\infty(\mathbb{D})$. In an effort to classify ideal membership for finitely-generated ideals in $H^\infty(\mathbb{D})$, Wolff [G] proved the following version:

Theorem A (Wolff). *If*

$$\begin{aligned} \{f_j\}_{j=1}^n \subset H^\infty(\mathbb{D}), H \in H^\infty(\mathbb{D}) \quad \text{and} \\ |H(z)| \leq \left(\sum_{j=1}^n |f_j(z)|^2 \right)^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{D}, \end{aligned} \tag{1}$$

then

$$H^3 \in \mathcal{I}(\{f_j\}_{j=1}^n),$$

the ideal generated by $\{f_j\}_{j=1}^n$ in $H^\infty(\mathbb{D})$.

It is known that (1) is not, in general, sufficient for H itself to be in $\mathcal{I}(\{f_j\}_{j=1}^n)$, see Rao [G]; or even for H^2 to be in $\mathcal{I}(\{f_j\}_{j=1}^n)$, see Treil [T].

Recall that if we consider the radical of the ideal, $\mathcal{I}(\{f_j\}_{j=1}^n)$, i.e.

$$\text{Rad}(\{f_j\}_{j=1}^n) \stackrel{\text{def}}{=} \{h \in H^\infty(\mathbb{D}) : \exists n \in \mathbb{N} \text{ with } h^n \in \mathcal{I}(\{f_j\}_{j=1}^n)\},$$

then (1) gives a characterization of radical ideal membership.

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That is:

Theorem B (Wolff). *Let $\{H, f_j : j = 1, \dots, n\} \subset H^\infty(\mathbb{D})$. Then $H \in \text{Rad}(\{f_j\}_{j=1}^n)$ if and only if there exists $C_0 < \infty$ and $m \in \mathbb{N}$ such that*

$$|H^m(z)| \leq C_0 \sum_{j=1}^n |f_j(z)|^2 \text{ for all } z \in \mathbb{D}.$$

For the algebra of multipliers on Dirichlet space, the analogue of the corona theorem was established in Tolokonnikov [To] and, for infinitely many generators, this was done in Trent [Tr2]. The purpose of this paper is to establish an analogue of Wolff's results, Theorems A and B, for the algebra of multipliers on Dirichlet space.

We use \mathcal{D} to denote the Dirichlet space on the unit disk, \mathbb{D} . That is,

$$\mathcal{D} = \{f : \mathbb{D} \rightarrow \mathbb{C} \mid f \text{ is analytic on } \mathbb{D} \text{ and for } f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$\|f\|_{\mathcal{D}}^2 = \sum_{n=0}^{\infty} (n+1) |a_n|^2 < \infty\}.$$

We will use other equivalent norms for smooth functions in \mathcal{D} as follows,

$$\begin{aligned} \|f\|_{\mathcal{D}}^2 &= \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_D |f'(z)|^2 dA(z) \quad \text{and} \\ \|f\|_{\mathcal{D}}^2 &= \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} d\sigma d\theta. \end{aligned}$$

Also, we will consider $\bigoplus_1^{\infty} \mathcal{D}$ as an l^2 -valued Dirichlet space. The norms in this case are exactly as above but we will replace the absolute value by l^2 -norms. Moreover, we use \mathcal{HD} to denote the harmonic Dirichlet space (restricted to the boundary of \mathbb{D}). The functions in \mathcal{D} have only vanishing negative Fourier coefficients, whereas the functions in \mathcal{HD} may have negative fourier coefficients which do not vanish. Again, if f is smooth on ∂D , the boundary of the unit disk D , then

$$\|f\|_{\mathcal{HD}}^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} d\sigma d\theta.$$

We use $\mathcal{M}(\mathcal{D})$ to denote the multiplier algebra of Dirichlet space, defined as: $\mathcal{M}(\mathcal{D}) = \{\phi \in \mathcal{D} : \phi f \in \mathcal{D} \text{ for all } f \in \mathcal{D}\}$, and we will denote the multiplier algebra of harmonic Dirichlet space by $\mathcal{M}(\mathcal{HD})$, defined similarly (but only on ∂D).

Given $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$, we consider $F(z) = (f_1(z), f_2(z), \dots)$ for $z \in D$. We define the row operator $M_F^R : \bigoplus_1^\infty \mathcal{D} \rightarrow \mathcal{D}$ by

$$M_F^R \left(\{h_j\}_{j=1}^\infty \right) = \sum_{j=1}^\infty f_j h_j \text{ for } \{h_j\}_{j=1}^\infty \in \bigoplus_1^\infty \mathcal{D}.$$

Similarly, define the column operator $M_F^C : \mathcal{D} \rightarrow \bigoplus_1^\infty \mathcal{D}$ by

$$M_F^C(h) = \{f_j h\}_{j=1}^\infty \text{ for } h \in \mathcal{D}.$$

We notice that \mathcal{D} is a reproducing kernel (r.k.) Hilbert space with r.k.

$$k_w(z) = \frac{1}{z\bar{w}} \log \left(\frac{1}{1 - z\bar{w}} \right) \text{ for } z, w \in \mathbb{D}$$

and it is well known (see [AM]) that

$$\frac{1}{k_w(z)} = 1 - \sum_{n=1}^\infty c_n (z\bar{w})^n, \quad c_n > 0, \text{ for all } n.$$

Hence, Dirichlet space has a reproducing kernel with “one positive square” or a “complete Nevanlinna-Pick” kernel. This property will be used to complete the first part of our proof.

An important relationship between the multipliers and reproducing kernels is that for $\phi \in \mathcal{M}(\mathcal{D})$ and $z \in \mathbb{D}$,

$$M_\phi^* k_z = \overline{\phi(z)} k_z.$$

This automatically implies that $\|\phi\|_\infty \leq \|M_\phi\|$, so $\mathcal{M}(\mathcal{D}) \subseteq H^\infty(\mathbb{D})$.

Similarly, if $\phi_{ij} \in \mathcal{M}(\mathcal{D})$ and $M_{[\phi_{ij}]_{j=1}^\infty} \in B(\bigoplus_1^\infty \mathcal{D})$, then for $\underline{x} \in l^2$ and $z \in \mathbb{D}$, we have

$$M_{[\phi_{ij}]}^* (\underline{x} k_z) = [\phi_{ij}(z)]^* (\underline{x} k_z).$$

Again, it follows that

$$\sup_{z \in D} \|\phi_{ij}(z)\|_{B(l^2)} \leq \|M_{[\phi_{ij}]}\|_{B(\bigoplus_1^\infty \mathcal{D})}$$

and so

$$\mathcal{M}(\bigoplus_1^\infty \mathcal{D}) \subseteq H_{B(l^2)}^\infty(\mathbb{D}).$$

It is clear that $\mathcal{M}(H^2(\mathbb{D})) = H^\infty(\mathbb{D})$ but $\mathcal{M}(\mathcal{D}) \subsetneq H^\infty(\mathbb{D})$ (e.g., $\sum_{n=1}^\infty \frac{z^{n^3}}{n^2}$ is in $H^\infty(\mathbb{D})$ but is not in \mathcal{D} and so not in $\mathcal{M}(\mathcal{D})$). Hence, $\mathcal{M}(\mathcal{D}) \subsetneq H^\infty(\mathbb{D}) \cap \mathcal{D}$.

Also, it is worthwhile to note that the pointwise hypothesis that $F(z) F(z)^* \leq 1$ for $z \in \mathbb{D}$, implies that the analytic Toeplitz operators T_F^R and T_F^C defined on $\bigoplus_1^\infty H^2(\mathbb{D})$ and $H^2(\mathbb{D})$ in analogy to that of M_F^R and M_F^C are bounded and

$$\|T_F^R\| = \|T_F^C\| = \sup_{z \in \mathbb{D}} \left(\sum_{j=1}^\infty |f_j(z)|^2 \right)^{\frac{1}{2}} \leq 1.$$

But, since $M(\mathcal{D}) \subsetneq H^\infty(\mathbb{D})$, the pointwise upperbound hypothesis will not be sufficient to conclude that M_F^R and M_F^C are bounded on Dirichlet space. However, $\|M_F^R\| \leq \sqrt{18} \|M_F^C\|$ from [Tr2]. Thus, we will replace the natural normalization that $F(z) F(z)^* \leq 1$ for all $z \in \mathbb{D}$, by the stronger condition that $\|M_F^C\| \leq 1$.

Then we have the following theorem:

Theorem 1. *Let $H, \{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$. Assume that*

$$(a) \|M_F^C\| \leq 1$$

$$\text{and } (b) |H(z)| \leq \sqrt{\sum_{j=1}^\infty |f_j(z)|^2} \text{ for all } z \in \mathbb{D}.$$

Then there exist $\{g_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$ with

$$\|M_G^C\| < \infty$$

$$\text{and } F G^T = H^3.$$

Of course, it should be noted that for only a finite number of multipliers, $\{f_j\}$, condition (a) of Theorem 1 can always be assumed, so we have the exact analogue of Wolff's theorem in the finite case.

First, let's outline the method of our proof. Assume that $F \in \mathcal{M}_{l^2}(\mathcal{D})$ and $H \in \mathcal{M}(\mathcal{D})$ satisfy the hypotheses (a) and (b) of Theorem 1. Then we show that there exists a constant $K < \infty$, so that

$$M_{H^3} M_{H^3}^* \leq K^2 M_F^R M_F^{\star R}. \quad (2)$$

Given (2), a commutant lifting theorem argument as it appears in, for example, Trent [Tr2], completes the proof by providing a $G \in \mathcal{M}_{l^2}(\mathcal{D})$, so that $\|M_G^C\| \leq K$ and $F G^T = H^3$.

But (2) is equivalent to the following: there exists a constant $K < \infty$ so that, for any $h \in \mathcal{D}$, there exists $\underline{u}_h \in \bigoplus_1^\infty \mathcal{D}$ such that

$$\begin{aligned} \text{(i)} \quad & M_F^R(\underline{u}_h) = H^3 h \quad \text{and} \\ \text{(ii)} \quad & \|\underline{u}_h\|_{\mathcal{D}} \leq K \|h\|_{\mathcal{D}}. \end{aligned} \tag{3}$$

Hence, our goal is to show that (3) follows from (a) and (b). For this we need a series of lemmas.

Lemma 1. *Let $\{c_j\}_{j=1}^\infty \in l^2$ and $C = (c_1, c_2, \dots) \in B(l^2, \mathbb{C})$. Then there exists Q such that the entries of Q are either 0 or $\pm c_j$ for some j and $CC^*I - C^*C = QQ^*$. Also, $\text{range of } Q = \text{kernel of } C$.*

We will apply this lemma in our case with $C = F(z)$ for each $z \in \mathbb{D}$, when $F(z) \neq 0$. A proof of a more general version can be found in Trent [Tr2].

Given condition (b) of Theorem 1 for all $z \in \mathbb{D}$, $F \in \mathcal{M}_{l^2}(\mathcal{D})$ and $H \in \mathcal{M}(\mathcal{D})$ with H being not identically zero, we lose no generality assuming that $H(0) \neq 0$. If $H(0) = 0$, but $H(a) \neq 0$, let $\beta(z) = \frac{a-z}{1-\bar{a}z}$ for $z \in \mathbb{D}$. Then since (b) holds for all $z \in \mathbb{D}$, it holds for $\beta(z)$. So we may replace H and F by $H\circ\beta$ and $F\circ\beta$, respectively. If we prove our theorem for $H\circ\beta$ and $F\circ\beta$, then there exists $G \in \mathcal{M}_{l^2}(\mathcal{D})$ so that $(F\circ\beta)G = H\circ\beta$ and hence $F(G\circ\beta^{-1}) = H$ and $G\circ\beta^{-1} \in \mathcal{M}_{l^2}(\mathcal{D})$, so we were done. Thus, we may assume that $H(0) \neq 0$ in (b), so $\|F(0)\|_2 \neq 0$. This normalization will let us apply some relevant lemmas from [Tr1].

It suffices to establish (i) and (ii) for any dense set of functions in \mathcal{D} , so we will use polynomials. First, we will assume F and H are analytic on $\mathbb{D}_{1+\epsilon}(0)$. In this case, we write the most general solution of the pointwise problem on $\bar{\mathbb{D}}$ and find an analytic solution with uniform bounds. Then we remove the smoothness hypotheses on F and H .

For a polynomial, h , we take

$$\underline{u}_h(z) = F(z)^* (F(z)F(z)^*)^{-1} H^3 h - Q(z)\underline{k}(z), \text{ where } \underline{k}(z) \in l^2 \text{ for } z \in \bar{\mathbb{D}}.$$

We have to find $\underline{k}(z)$ so that $\underline{u}_h \in \bigoplus_1^\infty \mathcal{D}$. Thus we want $\bar{\partial}_z \underline{u}_h = 0$ in \mathbb{D} .

Therefore, we will try

$$\underline{u}_h = \frac{F^* H^3 h}{F F^*} - Q \left(\frac{\widehat{Q^* F'^* H^3 h}}{(F F^*)^2} \right),$$

where \widehat{k} is the Cauchy transform of k on \mathbb{D} . Note that for k smooth on \mathbb{D} and $z \in \mathbb{D}$,

$$\widehat{k}(z) = -\frac{1}{\pi} \int_D \frac{k(w)}{w-z} dA(w) \quad \text{and} \quad \bar{\partial} \widehat{k}(z) = k(z) \quad \text{for } z \in \mathbb{D}.$$

See [A] for background on the Cauchy transform.

Then it's clear that $M_F^R(\underline{u}_h) = H^3 h$ and \underline{u}_h is analytic. Hence, we will be done in the smooth case if we are able to find $K < \infty$, independent of the polynomial, h , and $\epsilon > 0$, such that

$$\|\underline{u}_h\|_{\mathcal{D}} \leq K \|h\|_{\mathcal{D}} \quad (4)$$

Lemma 2. *Let \underline{w} be a harmonic function on $\overline{\mathbb{D}}$, then*

$$\int_D \|Q' \underline{w}\|_{l^2}^2 dA \leq 8 \|\underline{w}\|_{\mathcal{HD}}^2.$$

Proof. Let \underline{w} be a vector-valued harmonic function on $\overline{\mathbb{D}}$. Write $\underline{w} = \underline{x} + \underline{\bar{y}}$, where \underline{x} and $\underline{\bar{y}}$ are respectively the analytic and co-analytic parts of \underline{w} .

We have

$$\begin{aligned} \int_D \|Q' \underline{w}\|_{l^2}^2 dA &= \int_D \|Q' \underline{x} + Q' \underline{\bar{y}}\|_{l^2}^2 dA \\ &\leq 2 \int_D \|Q' \underline{x}\|_{l^2}^2 dA + 2 \int_D \|Q' \underline{\bar{y}}\|_{l^2}^2 dA. \end{aligned}$$

Now

$$\begin{aligned} \int_D \|Q' \underline{x}\|_{l^2}^2 dA &= \int_D \langle Q'^* Q' \underline{x}, \underline{x} \rangle_{l^2} dA \\ &\leq \int_D \langle F' F'^* \underline{x}, \underline{x} \rangle_{l^2} dA \\ &\leq \int_D \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\bar{f}'_j x_k|^2 dA \\ &\leq 2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_D |(f_j x_k)'|^2 dA + 2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_D |f_j x'_k|^2 dA \\ &\leq 2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \|M_{f_j} x_k\|_{\mathcal{D}}^2 + 2 \sum_{k=1}^{\infty} \|x_k\|_{\mathcal{D}}^2 dA \\ &\leq 2 \sum_{k=1}^{\infty} \|M_F^C\|_{\mathcal{D}}^2 \|x_k\|_{\mathcal{D}}^2 + 2 \sum_{k=1}^{\infty} \|x_k\|_{\mathcal{D}}^2 \\ &= 4 \|\underline{x}\|_{\mathcal{D}}^2. \end{aligned}$$

Similarly, we can show that $\int_D \|Q' \underline{y}\|_{l^2}^2 dA \leq 4 \|\underline{y}\|_{\mathcal{D}}^2$.

Thus,

$$\begin{aligned} \int_D \|Q' \underline{w}\|_{l^2}^2 dA &\leq 8 \|\underline{x}\|_{\mathcal{D}}^2 + 8 \|\underline{y}\|_{\mathcal{D}}^2 \\ &= 8 \|\underline{x} + \underline{y}\|_{\mathcal{HD}}^2 \\ &= 8 \|\underline{w}\|_{\mathcal{HD}}^2. \end{aligned}$$

□

Lemma 3. *Let the operator T be defined on $L^2(\mathbb{D}, dA)$ by*

$$(Tf)(\lambda) = \int_D \frac{f(z)}{(z - \lambda)(1 - z \bar{\lambda})} dA(z),$$

for $\lambda \in \mathbb{D}$ and $f \in L^2(\mathbb{D}, dA)$. Then

$$\|Tf\|_A^2 \leq 100 \pi^2 \|f\|_A^2.$$

Proof. To show that the singular integral operator, T , is bounded on $L^2(\mathbb{D}, dA)$, we apply Zygmund's method of rotations [Z] and apply Schur's lemma an infinite number of times.

Let $f(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} z^j \bar{z}^k$, where $a_{ij} = 0$ except for a finite number of terms. For $z = r e^{i\theta}$, we relabel, so that

$$f(r e^{i\theta}) = \sum_{l=-\infty}^{\infty} f_l(r) e^{il\theta}, \text{ where } f_l(r) = \sum_{k=0}^{\infty} a_{l+k, k} r^{l+2k}.$$

Then

$$\|f\|_A^2 = \sum_{l=-\infty}^{\infty} \|f_l(r)\|_{L^2[0,1]}^2,$$

where the measure on $L^2[0, 1]$ is " rdr ".

Now

$$\begin{aligned}
(Tf)(\lambda) &= \int_D \frac{f(z)}{(z-\lambda)(1-z\bar{\lambda})} dA(z) \\
&= \sum_{n=0}^{\infty} \bar{\lambda}^n \int_D \left[\frac{1}{z-\lambda} \right] z^n f(z) dA(z) \\
&= \sum_{n=0}^{\infty} \bar{\lambda}^n \left[\int_{|z|<|\lambda|} \frac{1}{z-\lambda} + \int_{|\lambda|<|z|} \frac{1}{z-\lambda} \right] z^n f(z) dA(z) \\
&= \sum_{n=0}^{\infty} \bar{\lambda}^n \left[\frac{1}{-\lambda} \int_{|z|<|\lambda|} \sum_{p=0}^{\infty} \frac{z^p}{\lambda^p} + \int_{|\lambda|<|z|} \frac{1}{z} \sum_{p=0}^{\infty} \frac{\lambda^p}{z^p} \right] z^n f(z) dA(z) \\
&= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (-1) \bar{\lambda}^n \frac{1}{\lambda} \int_{|z|<|\lambda|} \frac{z^{n+p}}{\lambda^p} \left(\sum_{l=-\infty}^{\infty} f_l(r) e^{il\theta} \right) dA(z) \\
&\quad + \sum_{n=0}^{\infty} \bar{\lambda}^n \sum_{p=0}^{\infty} \int_{|\lambda|<|z|} \frac{\lambda^p}{z^{p+1}} z^n \left(\sum_{l=-\infty}^{\infty} f_l(r) e^{il\theta} \right) dA(z).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(Tf)(se^{it}) &= \\
&\sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (-1) s^n e^{-int} \frac{e^{-it}}{s} \int_{-\pi}^{\pi} \int_0^s \frac{r^{n+p} e^{i(n+p+l)\theta}}{s^p e^{ipt}} f_l(r) r dr d\theta \\
&+ \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} s^n e^{-int} \sum_{p=0}^{\infty} \int_{-\pi}^{\pi} \int_s^1 \frac{s^p e^{ipt}}{r^{p+1} e^{i(p+1)\theta}} r^n e^{in\theta} e^{il\theta} f_l(r) r dr d\theta. \quad (\star)
\end{aligned}$$

Taking $l = 0$ in (\star) , we get that

$$\begin{aligned}
(Tf_0)(se^{it}) &= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (-1) s^n e^{-int} \frac{e^{-it}}{s} \int_{-\pi}^{\pi} \int_0^s \frac{r^{n+p} e^{i(n+p)\theta}}{s^p e^{ipt}} f_0(r) r dr d\theta \\
&+ \sum_{n=0}^{\infty} s^n e^{-int} \sum_{p=0}^{\infty} \int_{-\pi}^{\pi} \int_s^1 \frac{s^p e^{ipt}}{r^{p+1} e^{i(p+1)\theta}} r^n e^{in\theta} f_0(r) r dr d\theta.
\end{aligned}$$

Simplifying the above,

$$\begin{aligned}
(Tf_0)(se^{it}) &= -2\pi \int_0^1 \chi_{(0,s)}(r) \frac{f_0(r) e^{-it}}{s} r dr \\
&\quad + 2\pi s e^{-it} \sum_{n=0}^{\infty} s^{2n} \int_0^1 \chi_{(s,1)}(r) f_0(r) r dr.
\end{aligned}$$

So

$$(Tf_0)(se^{it}) = 2\pi e^{-it}(T_0f_0)(s),$$

where we define T_0 on $L^2([0, 1], r dr)$ by

$$(T_0f_0)(s) = -\int_0^1 \chi_{(0,s)}(r) \left(\frac{r}{s}\right) f_0(r) dr + \frac{s}{1-s^2} \int_0^1 \chi_{(s,1)}(r) f_0(r) r dr.$$

A similar calculation shows that when $l \geq 1$, then

$$(Tf_l(r) e^{il\theta})(se^{it}) = 2\pi e^{i(l-1)t}(T_lf_l)(s),$$

where

$$(T_lf_l)(s) = \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r) \left(\frac{s}{r}\right)^{l-1} f_0(r) r dr.$$

Similarly, when $l < 0$,

$$(Tf_l(r) e^{il\theta})(se^{it}) = 2\pi e^{i(l-1)t}(T_lf_l)(s),$$

where

$$\begin{aligned} (T_lf_l)(s) = & -\left(\sum_{n=0}^{-l} s^{2n}\right) \int_0^1 \chi_{(0,s)}(r) \left(\frac{r}{s}\right)^{1-l} f_l(r) dr \\ & + \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r) (rs)^{1-l} f_l(r) dr. \end{aligned}$$

Hence,

$$(Tf)(se^{it}) = 2\pi \sum_{l=-\infty}^{\infty} e^{i(l-1)t}(T_lf_l)(s),$$

$$\text{for } (T_lf_l)(s) = \begin{cases} -\left(\sum_{n=0}^{-l} s^{2n}\right) \int_0^1 \chi_{(0,s)}(r) \left(\frac{r}{s}\right)^{1-l} f_l(r) dr \\ \quad + \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r) (rs)^{1-l} f_l(r) dr & \text{for } l \leq 0 \\ \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r) \left(\frac{s}{r}\right)^{l-1} f_0(r) r dr & \text{for } l > 0. \end{cases}$$

By our construction,

$$\|Tf\|_A^2 = 4\pi^2 \sum_{l=-\infty}^{\infty} \|T_lf_l\|_{L^2[0,1]}^2,$$

where the measure on $L^2[0, 1]$ is “ $r dr$ ”. Thus to prove our lemma, it suffices to prove that

$$\sup_l \|T_l\|_{B(L^2[0,1], L^2[0,1])} \leq 5 < \infty. \quad (\star\star)$$

Once we prove $(\star\star)$, we can conclude that

$$\|Tf\|_A^2 \leq 100 \pi^2 \sum_{l=-\infty}^{\infty} \|f_l\|_{L^2[0,1]}^2 = 100 \pi^2 \|f\|_A^2.$$

For the case $l = 0$, we get that

$$\begin{aligned} & \int_0^1 |(T_0 f_0)(se^{it})|^2 s ds \\ & \leq 2 \int_0^1 \left| - \int_0^1 \chi_{(0,s)}(r) \frac{f_0(r) - e^{-it}}{s} r dr \right|^2 s ds \\ & \quad + 2 \int_0^1 \left| \frac{s}{1-s^2} (e^{-it}) \int_0^1 \chi_{(s,1)}(r) f_0(r) r dr \right|^2 s ds \\ & \leq 2 \int_0^1 \frac{1}{s^2} \left[\int_0^1 \chi_{(0,s)}(u) |f_0(u)| u du \int_0^1 \chi_{(0,s)}(v) |f_0(v)| v dv \right] s ds \\ & \quad + 2 \int_0^1 \frac{s^2}{(1-s^2)^2} \left[\int_0^1 \chi_{(s,1)}(x) |f_0(x)| x dx \int_0^1 \chi_{(s,1)}(y) |f_0(y)| y dy \right] s ds. \end{aligned}$$

Let's take the first term, which is

$$\begin{aligned} & \int_0^1 \int_0^1 |f_0(u)| |f_0(v)| \left(\int_0^1 \chi_{(0,s)}(u) \chi_{(0,s)}(v) \frac{ds}{s} \right) u du v dv \\ & = \int_0^1 \int_0^1 |f_0(u)| |f_0(v)| \ln \left(\frac{1}{\max\{u, v\}} \right) u du v dv. \end{aligned}$$

By Schur's Test we know that

$$\int_0^1 \int_0^1 |f_0(u)| |f_0(v)| \ln \left(\frac{1}{\max\{u, v\}} \right) u du v dv \leq C_1^2 \int_0^1 |f(u)|^2 u du$$

if and only if there exist $p \geq 0$ a.e. and $C_1 < \infty$, satisfying

$$\int_0^1 \ln \left(\frac{1}{\max\{u, v\}} \right) p(u) u du \leq C_1 p(v)$$

for a.e. $v \in [0, 1]$.

We take $p(u) = 1$. Now

$$\int_0^v \ln \left(\frac{1}{v} \right) u du = \frac{1}{4} \frac{\ln \left(\frac{1}{v^2} \right)}{\left(\frac{1}{v^2} \right)} \leq \frac{1}{4}$$

and

$$\int_v^1 \ln \left(\frac{1}{u} \right) u du \leq 1 - v \leq 1.$$

Therefore,

$$C_1 = \frac{1}{4} + 1 = \frac{5}{4}.$$

Again, we will repeat the same argument for the second term,

$$\int_0^1 \int_0^1 |f_0(x)| |f_0(y)| \left[\int_0^{\min\{x,y\}} \frac{s^2}{(1-s^2)^2} s ds \right] x dx y dy.$$

Changing variables and computing, we get that

$$\begin{aligned} & \int_0^1 \int_0^1 |f_0(x)| |f_0(y)| \left[\int_0^{\min\{x,y\}} \frac{s^2}{(1-s^2)^2} s ds \right] x dx y dy \\ & \leq \int_0^1 \int_0^1 |f_0(x)| |f_0(y)| \left[\frac{1}{2} \frac{1}{(1 - (\min\{x,y\})^2)} \right] x dx y dy. \end{aligned}$$

For this second term, we take $p(x) = \frac{1}{\sqrt{1-x^2}}$. Therefore,

$$\int_0^y \frac{1}{2} \frac{1}{1-x^2} \frac{1}{\sqrt{1-x^2}} x dx \leq \frac{1}{2} \frac{1}{\sqrt{1-y^2}}.$$

Also,

$$\int_y^1 \frac{1}{2} \frac{1}{1-y^2} \frac{1}{\sqrt{1-x^2}} x dx = \frac{1}{2} \frac{1}{\sqrt{1-y^2}}.$$

So we get $C_2 \leq 1$. Hence

$$\int_0^1 |(T_0 f_0)(s)|^2 s ds \leq \frac{9}{2} \int_0^1 |f_0(u)|^2 u du.$$

Applying Schur's Test for $l \geq 1$ with $p(u) = \frac{1}{\sqrt{1-u^2}}$, we get the estimate $C_l \leq \frac{3}{2}$, independent of l . Similarly, for $l < 0$ with $p(u) = 1$ and $p(u) = \frac{1}{\sqrt{1-u^2}}$ for each of the two terms, respectively, we get the estimate $C_l \leq 5$, independent of l . Thus we conclude that

$$\sup_l \|T_l\|_{B(L^2[0,1], L^2[0,1])} \leq 5.$$

This finishes the proof of Lemma 3. \square

Lemma 4. *If Q is a multiplier of \mathcal{D} , then*

$$(1 - |z|^2) |Q'(z)| \leq \|M_Q\|_{B(\mathcal{D})} \text{ for all } z \in \mathbb{D}.$$

Proof. Define $\varphi : D \rightarrow D$ as $\varphi(z) = \frac{Q(z)}{\|M_Q\|}$ for all $z \in \mathbb{D}$. Now use the Schwarz lemma and the fact that $\|\varphi\|_{\infty, \mathbb{D}} \leq \|M_\varphi\|$ to complete the proof. \square

We are now ready to prove Theorem 1.

Proof. First, we will prove the theorem for smooth functions on $\overline{\mathbb{D}}$ and get a uniform bound. Then we will remove the smoothness hypothesis.

Assume that (a) and (b) of Theorem 1 hold for F and H and that F and H are analytic on $\mathbb{D}_{1+\epsilon}(0)$.

Then our main goal is to show that there exists a constant $K < \infty$, independent of ϵ , so that for any polynomial, h , there exists $\underline{u}_h \in \bigoplus_1^\infty \mathcal{D}$ such that $M_F^R(\underline{u}_h) = H^3 h$ and $\|\underline{u}_h\|_{\mathcal{D}}^2 \leq K \|h\|_{\mathcal{D}}^2$.

We take $\underline{u}_h = \frac{F^* H^3 h}{F F^*} - Q \left(\widehat{\frac{Q^* F'^* H^3 h}{(F F^*)^2}} \right)$. Then \underline{u}_h is analytic and $M_F^R(\underline{u}_h) = H^3 h$.

We know that

$$\|\underline{u}_h\|_{\mathcal{D}}^2 = \int_{-\pi}^{\pi} \|\underline{u}_h(e^{it})\|^2 d\sigma(t) + \int_D \|(\underline{u}_h(z))'\|^2 dA(z).$$

Condition (b) implies that

$$\int_{-\pi}^{\pi} \left\| \frac{F^* H^3 h}{F F^*} - Q \left(\widehat{\frac{Q^* F'^* H^3 h}{(F F^*)^2}} \right) \right\|^2 d\sigma(t) \leq C_0^2 \|h\|_{\sigma}^2,$$

where C_0 can be chosen to be 15 (See [Tr1]). Hence, we only need to show that

$$\int_D \left\| \left(\frac{F^* H^3 h}{F F^*} - Q \left(\widehat{\frac{Q^* F'^* H^3 h}{(F F^*)^2}} \right) \right)' \right\|^2 dA(z) \leq C^2 \|h\|_{\mathcal{D}}^2$$

for some $C < \infty$.

Now

$$\begin{aligned}
& \int_D \left\| \left(\frac{F^* H^3 h}{F F^*} - Q \left(\frac{\widehat{Q^* F'^* H^3 h}}{(F F^*)^2} \right) \right)' \right\|^2 dA(z) \\
& \leq 2 \int_D \left\| \left(\frac{F^* H^3 h}{F F^*} \right)' \right\|^2 dA(z) + 2 \int_D \left\| \left(Q \left(\frac{\widehat{Q^* F'^* H^3 h}}{(F F^*)^2} \right) \right)' \right\|^2 dA(z) \\
& \leq 4 \underbrace{\int_D \left\| \frac{F^* 3 H^2 H' h}{F F^*} \right\|^2 dA(z)}_{(a')} + 8 \underbrace{\int_D \left\| \frac{F^* H^3 h'}{F F^*} \right\|^2 dA(z)}_{(b')} \\
& \quad + 8 \underbrace{\int_D \left\| \frac{F^* H^3 h' F' F^*}{(F F^*)^2} \right\|^2 dA(z)}_{(c')} + 4 \underbrace{\int_D \left\| Q' \left(\frac{\widehat{Q^* F'^* H^3 h}}{(F F^*)^2} \right) \right\|^2 dA(z)}_{(d')} \\
& \quad + 4 \underbrace{\int_D \left\| Q \left(\frac{\widehat{Q^* F'^* H^3 h}}{(F F^*)^2} \right)' \right\|^2 dA(z)}_{(e')}.
\end{aligned}$$

Then

$$\begin{aligned}
(a') &= \int_D \left\| \frac{F^* 3 H^2 H' h}{F F^*} \right\|^2 dA(z) = 9 \int_D \left\| \frac{F^*}{\sqrt{F F^*}} \frac{H}{\sqrt{F F^*}} H H' h \right\|^2 dA(z) \\
&\leq 9 \int_D \|H' h\|^2 dA(z) \\
&\leq 18 (\|M_H\|^2 + \|H\|_\infty^2) \|h\|_{\mathcal{D}}^2 \\
&\leq 36 \|M_H\|^2 \|h\|_{\mathcal{D}}^2. \\
(b') &= \int_D \left\| \frac{F^* H^3 h'}{F F^*} \right\|^2 dA(z) \leq \int_D \|h'\|^2 dA(z) \leq \|h\|_{\mathcal{D}}^2. \\
(c') &= \int_D \left\| \frac{F^* H^3 h F' F^*}{(F F^*)^2} \right\|^2 dA(z) = \int_D \left\| \frac{F^* F' F^*}{\sqrt{F F^*}} \frac{H^2}{F F^*} \frac{H}{\sqrt{F F^*}} h \right\|^2 dA(z) \\
&\leq \int_D \left\| \frac{F^* F' F^*}{\sqrt{F F^*}} h \right\|^2 dA(z) \\
&\leq \int_D \|F'^* h\|^2 dA(z) \leq 4 \|h\|_{\mathcal{D}}^2.
\end{aligned}$$

We use condition (a) of the theorem and the boundedness of the Beurling transform on $L^2(\mathbb{D}, dA)$ (with bound 14) to conclude that

$$(e') \leq 56(14)^2 \int_D \|F'^* h\|^2 dA \leq 224(14)^2 \|h\|_{\mathcal{D}}^2.$$

So we only need estimate (d') . For this, we have

$$\int_D \|Q' \left(\frac{Q^* \widehat{F'^* H^3 h}}{(FF^*)^2} \right)\|^2 dA(z) = \int_D \|Q' \widehat{w}\|^2 dA(z),$$

where $\widehat{w} = \left(\frac{Q^* \widehat{F'^* H^3 h}}{(FF^*)^2} \right)$ is a smooth function on $\overline{\mathbb{D}}$.

Therefore,

$$\int_D \|Q' \widehat{w}\|^2 dA(z) \leq 2 \underbrace{\int_D \|Q' \widehat{w} - Q' \widetilde{w}\|^2 dA(z)}_{(\alpha)} + 2 \int_D \|Q' \widetilde{w}\|^2 dA(z),$$

where $\widetilde{w}(z) = \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \widehat{w}(e^{it}) d\sigma(t)$ is the harmonic extension of \widehat{w} from $\partial\mathbb{D}$ to \mathbb{D} .

Lemma (2) tells us that

$$\int_D \|Q' \widetilde{w}\|^2 dA(z) \leq 8 \|\widetilde{w}\|_{\mathcal{HD}}^2.$$

Also, a lemma of [Tr2] implies that

$$\|\widetilde{w}\|_{\mathcal{HD}}^2 \leq \|w\|_A^2 + \|\widehat{w}\|_{\sigma}^2.$$

But, as we showed above

$$\|w\|_A^2 = \int_D \left\| \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right\|^2 dA(z) \leq \int_D \|F'^* h\|^2 dA(z) \leq 4 \|h\|_{\mathcal{D}}^2$$

and

$$\|\widehat{w}\|_{\sigma}^2 = \int_{-\pi}^{\pi} \left\| \left(\frac{Q^* \widehat{F'^* H^3 h}}{(FF^*)^2} \right) \right\|^2 d\sigma(t) \leq 15 \|h\|_{\sigma}^2.$$

from [Tr2].

Thus,

$$\int_D \|Q' \widetilde{w}\|^2 dA(z) \leq 8 [4 \|h\|_{\mathcal{D}}^2 + 15 \|h\|_{\sigma}^2].$$

Now we are just left with estimating (α) . We will use Lemmas 3 and 4. We have

$$\begin{aligned} (\alpha) &= \int_D \|Q' \widehat{w} - Q' \widetilde{w}\|^2 dA(z) \\ &= \int_D \|Q' \left[-\frac{1}{\pi} \int_D \frac{w(u)}{u-z} dA(u) - \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \widehat{w}(e^{it}) d\sigma(t) \right]\|^2 dA(z) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi^2} \int_D \|Q' \int_D w(u) \left[\frac{1}{u-z} \right. \\
&\quad \left. + \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} e^{-it} \frac{1}{1-ue^{-it}} d\sigma(t) \right] dA(u)\|^2 dA(z) \\
&= \frac{1}{\pi^2} \int_D \|Q' \int_D w(u) \left[\frac{1}{u-z} + \frac{\bar{z}}{1-u\bar{z}} \right] dA(u)\|^2 dA(z) \\
&= \frac{1}{\pi^2} \int_D \|Q' \int_D w(u) \left[\frac{1-|z|^2}{(u-z)(1-u\bar{z})} \right] dA(u)\|^2 dA(z) \\
&= \frac{1}{\pi^2} \int_D \|Q'(z) (1-|z|^2) T(w)(z)\|^2 dA(z) \\
&\leq \frac{\|M_Q\|^2}{\pi^2} \|T(w)\|_A^2 \text{ by Lemma 4} \\
&\leq 100 \pi^2 \frac{\|M_Q\|^2}{\pi^2} \|w\|^2 \text{ by Lemma 3} \\
&\leq 100 \|M_Q\|^2 \|h\|_{\mathcal{D}}^2 \leq 1800 \|h\|_{\mathcal{D}}^2.
\end{aligned}$$

Combining all these pieces, we see that in the smooth case

$$\|\underline{u}_h\|_{\mathcal{D}}^2 \leq K \|h\|_{\mathcal{D}}^2$$

for some constant $K < \infty$, which is independent of h and $\epsilon > 0$.

By the proof of Theorem 1 in the smooth case, we have

$$M_{F_r}^R (M_{F_r}^R)^* \leq K^2 M_{H_r} M_{H_r}^* \text{ for } 0 \leq r < 1.$$

Using a commutant lifting argument, there exists $G_r \in \mathcal{M}(\mathcal{D}, \bigoplus_1^{\infty} \mathcal{D})$ so that $M_{F_r}^R M_{G_r}^C = M_{H_r^3}$ and $\|M_{G_r}^R\| \leq K$. Then $M_{F_r}^R \rightarrow M_F^R$ and $M_{H_r} \rightarrow M_H$ as $r \uparrow 1$ in the \star -strong topology.

By compactness, we may choose a net with $G_{r_\alpha}^* \rightarrow G^*$ as $r_\alpha \rightarrow 1^-$. Since the multiplier algebra (as operators) is WOT closed, $G \in \mathcal{M}(\mathcal{D}, \bigoplus_1^{\infty} \mathcal{D})$. Also, since $F_{r_\alpha}^* \xrightarrow{s} F^*$, we get $M_{H_r}^* = M_{G_r}^{*C} M_{F_r}^{*R} \xrightarrow{WOT} M_G^{*C} M_F^{*R}$ and so $M_F^R M_G^C = M_{H^3}$ with entries of G in $\mathcal{M}(\mathcal{D})$ and $\|M_G^C\| \leq K$.

It might be of some interest to note that the norm of the operator, $\|M_G^C\|$, doesn't exceed $\sqrt{144\|M_H\|^2 + 73, 104}$.

This ends our proof. \square

Just as Wolff gets Theorem B for free, we get

Theorem 2. *Let $\{H, f_j : j = 1, \dots, n\} \subset \mathcal{M}(\mathcal{D})$. Then $H \in \text{Rad}(\{f_j\}_{j=1}^n)$ if and only if there exist $C_0 < \infty$ and $m \in \mathbb{N}$ such that*

$$|H^m(z)| \leq C_0 \sum_{j=1}^n |f_j(z)|^2 \text{ for all } z \in \mathbb{D}.$$

This paper discusses when H^3 belongs to $\mathcal{I}(\{f_j\}_{j=1}^n)$, the ideal generated by $\{f_j\}_{j=1}^n$ in $\mathcal{M}(\mathcal{D})$ and characterizes membership in the radical of the ideal, $\mathcal{I}(\{f_j\}_{j=1}^n)$. The question of strong sufficient conditions for H itself to belong to $\mathcal{I}(\{f_j\}_{j=1}^n)$ is more subtle. The first author has obtained some interesting results in this direction.

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